Nico Bakker's duck plane-filling curve

Herman Haverkort, 9 December 2022

There are many plane-filling curves—see, for example, Ventrella's book [1]. One may sample such a curve at regular intervals, that is, such that the same amount of area is filled between any pair of consecutive sample points, and then connect the sample points to obtain a sketch of the curve. Most published plane-filling curves can be sampled in such a way, that the resulting sketch follows the edges of a regular triangular or square grid, even if the curve fills a complicated fractal shape in the end. Most published plane-filling curves that cannot be sketched in this way fill simple shapes such as triangles or quadrilaterals. I only know of a few published plane-filling curves that are not sketched on a regular square or triangular grid and do fill fractal shapes:

- plane-filling curves that fill the Rauzy triangle fractal [2], or a part of it;
- plane-filling curves that fill the BMTV triangle fractal [3], or a part of it;
- and since very recently, Nico Bakker's *duck curve* and *Titanic* curves [4].

Bakker's duck curve is so called because one natural way of sketching it results in a wonderful tessellation of duck-shaped tiles of different sizes that fit perfectly together—for details, see Bakker's manuscript [4]. Below, however, we will not discuss the ducks, but focus on understanding the plane-filling character of the curve.

Definition of the duck curve

Consider a grid G_0, not of triangles or squares, but of parallelograms with side length ratio $\sqrt{2}$ and acute angles $\arccos(\frac{1}{4}\sqrt{2}) = \arctan(\sqrt{7})$, see Figure 1(a). These parallelograms have short diagonals as long as their longest sides, and long diagonals that are exactly $\sqrt{2}$ as long (and thus, twice as long as their shortest sides). To obtain a refined grid G_1, we replace each long edge in the grid by a sequence of three edges as illustrated in Figure 1(b). This results in a grid of parallelograms of the same shape as before, but rotated and scaled down by a factor $\sqrt{2}$, see Figure 1(c). Each short edge in the original grid becomes a long edge in the refined grid. Note that in the resulting grid, all edges can be traced back to a unique edge in the original grid, as indicated by the colours in Figure 1(c). If we repeat this refinement procedure ad infinitum, we obtain grids $G_2, G_3, ...$

When we start with an edge of G_0, we thus obtain sequences of edges $\sigma_0, \sigma_1, \sigma_2, ...$, where σ_0 is a single edge of G_0. For each $i > 0$, the sequence σ_i consists of edges of G_i and is obtained from σ_{i-1} by substituting, for each edge of σ_{i-1}, its replacement of one or three edges in G_i. We execute these substitutions such that the three edges that replace an edge are inserted and directed such, that they form a chain that has the same starting point and the same end point as the edge that is replaced by it. Thus, each σ_i constitutes a continuous path.

As i tends to ∞, the sequence σ_i ultimately converges to a copy of some fractal curve, see Figure 1(f). This is the duck curve. A duck curve can be more precisely described as a function f from $[0, 1]$ to points in the plane, as follows. Let e_0 be an edge in grid G_0 from the point $(0, 0)$ to the point $(1, 0)$; we will define f for the duck curve with these endpoints. When e is refined, it is replaced by three edges $\alpha(e_0)$, $\beta(e_0)$ and $\gamma(e_0)$, where α, β and γ are similarity transformations $\mathbb{R}^2 \to \mathbb{R}^2$ that shrink, rotate and translate the plane. Note that α has $(0, 0)$ as its fixed point, while γ has $(1, 0)$ as its fixed point. Each of the edges $\alpha(e_0)$, $\beta(e_0)$ and $\gamma(e_0)$ induces a smaller duck curve between its endpoints. To obtain a measure-preserving plane-filling curve, the duck curves between the endpoints of $\alpha(e_0)$, $\beta(e_0)$ and $\gamma(e_0)$ must cover the first quarter, the middle half, and the last quarter of the parameter range $[0, 1]$, respectively. The point $f(t)$ at position $t \in [0, 1]$ along the duck curve between the end points of e_0 must therefore satisfy:

$$f(t) = \begin{array}{ll} \alpha\left(f\left(4\,t\right)\right) & \text{if } t \in [0\;\;\;, 1/4] \\ \beta\left(f\left(2(t-1/4)\right)\right) & \text{if } t \in [1/4, 3/4] \\ \gamma\left(f\left(4(t-3/4)\right)\right) & \text{if } t \in [3/4, 1\;\;\;] \end{array}$$

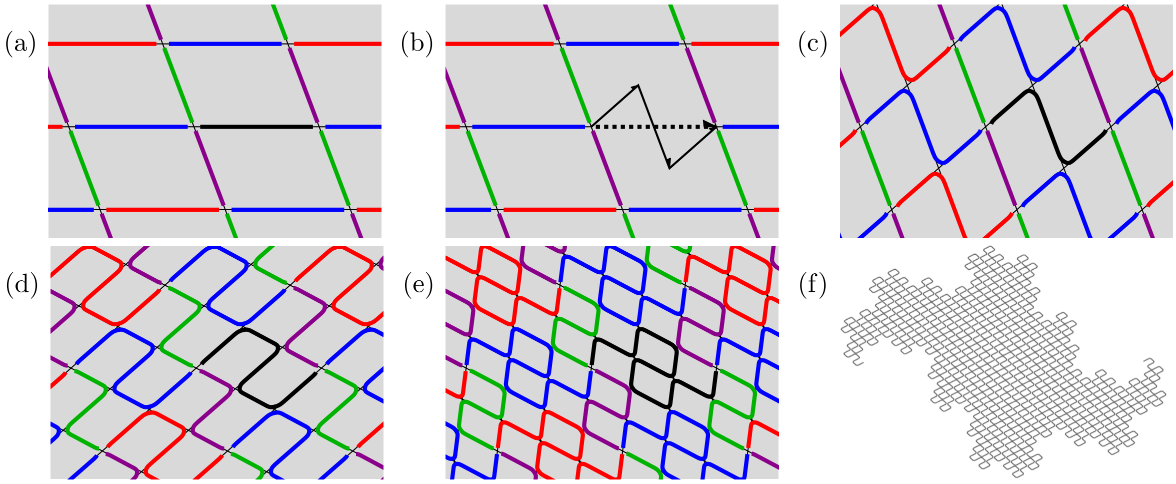

Figure 1: (a) Grid of parallelograms. (b) Replacing one edge of the grid. (c-e) The refined grids G_1, G_2, G_3. (f) A sketch of σ_i for high i.

Note that if $t \in \{1/4, 3/4\}$, two lines of this definition apply. We will now argue that the function f is well-defined and continuous and is thus a curve, and moreover, that it is a plane-filling curve.

Proof that the duck curve fills the plane

Bounding shape The first thing we need for our proof, is a coarse bounding shape for the image of the function f, that is, the part of the plane that is filled by the duck curve between the end points of e_0. From the above definition of f we learn that the image $\mathcal{I}$ of f must satisfy:

$$\mathcal{I} = \alpha(\mathcal{I}) \cup \beta(\mathcal{I}) \cup \gamma(\mathcal{I}).$$

Thus, $\mathcal{I}$ is the so-called *attractor* of an *iterated function system* that consists of the contractive maps α, β and γ. Under these conditions it can be proven[1] that $\mathcal{I}$ is contained in any compact set R such that $R \supseteq \alpha(R) \cup \beta(R) \cup \gamma(R)$. For this purpose we will take the closed ellipse R whose whose focal points are the end points of the duck curve and whose major axis is as long as $\sqrt{2}$ times the distance between the focal points, see Figure 2(a). We will call this ellipse the *range* of e_0. The range of a smaller edge $\lambda(e_0)$, that is obtained by applying a similarity transformation λ to e_0, is now $\lambda(R)$.

Function values Now we can have a look at the value of $f(t)$. First suppose $t = 0$. Then the recursive definition of $f(t)$ given above gives us $f(0) = \alpha(f(0))$. Since the only fixed point of α is $(0, 0)$, this implies $f(0) = (0, 0)$. By a similar analysis, we find $f(1) = (1, 0)$, and thus, $f(1/4) = \alpha(f(1)) = \beta(f(0))$ is the point where $\alpha(e_0)$ meets $\beta(e_0)$, and $f(3/4) = \beta(f(1)) = \gamma(f(0))$ is the point where $\beta(e_0)$ meets $\gamma(e_0)$. Now suppose we start with some value $t \in (0, 1)$ and expand the recursion, obtaining a sequence of the form:

$$f(t) = \lambda_1(f(t_1)) = \lambda_2(f(t_2)) = ...,$$

where each λ_i is a composition of similarity transformations from α, β and γ, and $\lambda_i(e_0)$ is an edge of G_i. Two things can happen: either we find that $f(t) = \lambda_i(f(t_i))$ for some i while $t_i \in \{1/4, 3/4\}$, or this never happens, no matter how far we expand the recursion. In the first case, the value of $\lambda_i(f(t_i))$ is well-defined, as we have seen above. In the second case, in every step of the recursion, the expansion of the recursion is unambiguous: each edge $\lambda_i(e_0)$ is either $\lambda_{i-1}(\alpha(e_0))$, $\lambda_{i-1}(\beta(e_0))$, or $\lambda_{i-1}(\gamma(e_0))$, and thus has a range that lies inside that of $\lambda_{i-1}(e_0)$ and whose diameter is smaller by at least a factor $\sqrt{2}$. Thus, the ranges of $e_0, \lambda_1(e_0), \lambda_2(e_0), ...$ converge to a point, and this point is $f(t)$.

Continuity A function f is continuous on $[0, 1]$ if $\lim_{t' \uparrow t} f(t') = f(t)$ for $t \in (0, 1]$, and $\lim_{t' \downarrow t} f(t') = f(t)$ for $t \in [0, 1)$. If, in the expansion of the recursion for $f(t)$, we never get $t_i \in \{1/4, 3/4\}$, the conditions

1. See, for example, Rice [5]: the proof for ball that is given there also works for arbitrary compact sets R.

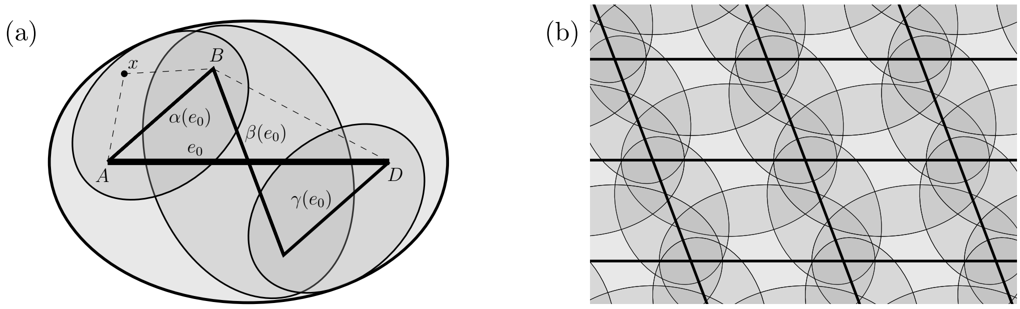

Figure 2: (a) Bounding ellipse of the duck curve. The bounding ellipse of e_0 is the set of points x such that the distance $||A-x|| + ||x-D||$ from A via x to D is at most $\sqrt{2}$. The bounding ellipse of $\alpha(e_0)$ is the set of points x such that $||A-x|| + ||x-B||$ is at most $\frac{1}{2}\sqrt{2}$. In the latter case, we have, by the triangle equality, $||A-x|| + ||x-D|| \leq ||A-x|| + ||x-B|| + ||B-D||$ and, therefore, $||A-x|| + ||x-D|| \leq (||A-x|| + ||x-B||) + ||B-D|| \leq \frac{1}{2}\sqrt{2} + \frac{1}{2}\sqrt{2} = \sqrt{2}$. Thus, any point x in the range of $\alpha(e_0)$ also lies in the range of e_0. (b) Bounding ellipses of edges of a grid. Each region's shade indicates the number of ellipses that covers the region, from 1 to 4. Each corner of a grid cell lies only in the bounding ellipses for the four edges that meet in that corner. Each grid cell is only intersected by the twelve different ellipses that contain one or two of its corners.

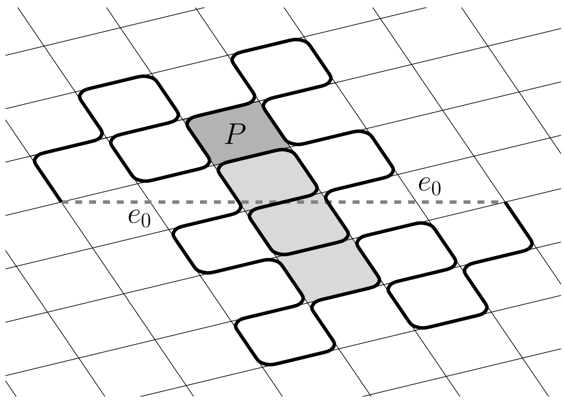

Figure 3: The edges of the fifth-level expansion σ_5 in G_5 of a single edge $\sigma_0 = \{e_0\}$ in G_0.

for continuity are obviously fulfilled: as t' gets closer to t, the expansions of the recursions for t and t' will share an ever larger initial sequence of edges $e_0, \lambda_1(e_0), \lambda_2(e_0), ...$ and corresponding nested bounding ellipses, and thus, $f(t')$ will converge to $f(t)$. If, at some point, we have $t_i = 1/4$, then, to verify continuity, we have to expand the recursion for $\lambda_i(f(t_i))$ with $\lambda_{i+1} = \lambda_i \circ \alpha$ and $t_{i+1} = 1$ to verify the first condition, whereas we expand the recursion with $\lambda_{i+1} = \lambda_i \circ \beta$ and $t_{i+1} = 0$ to verify the second condition; from there, we can complete the argument as before. The case of $t_i = 3/4$ can be handled in a similar manner. Thus, f is continuous.

Filling the plane To confirm that the duck curve is a plane-filling curve, all that remains to verify, is that its image $\mathcal{I}$ in the plane has non-zero Jordan measure, that is, the duck curve contains at least some points that are not on the boundary of $\mathcal{I}$. This can be confirmed as follows—the proof technique used is essentially an adaptation of Lévy's proof for Cesaro's curve [6] (pages 271–279).

It is easy to verify that the fifth level of refinement of the duck curve between the endpoints of e_0 contains all twelve edges incident on a particular grid cell P in G_5 (In fact, there are four such grid cells—see Figure 3). Now consider any point p in the interior of P. For any $i \geq 5$, let P_i be the cell (parallelogram) of G_i that contains p, where $P_5 = P$, and let E_i be the edges of G_i whose ranges contain p. Note (see Figure 2(b)) that E_i consists of at least one and at most four of the twelve edges that have at least one vertex on the boundary of P_i.

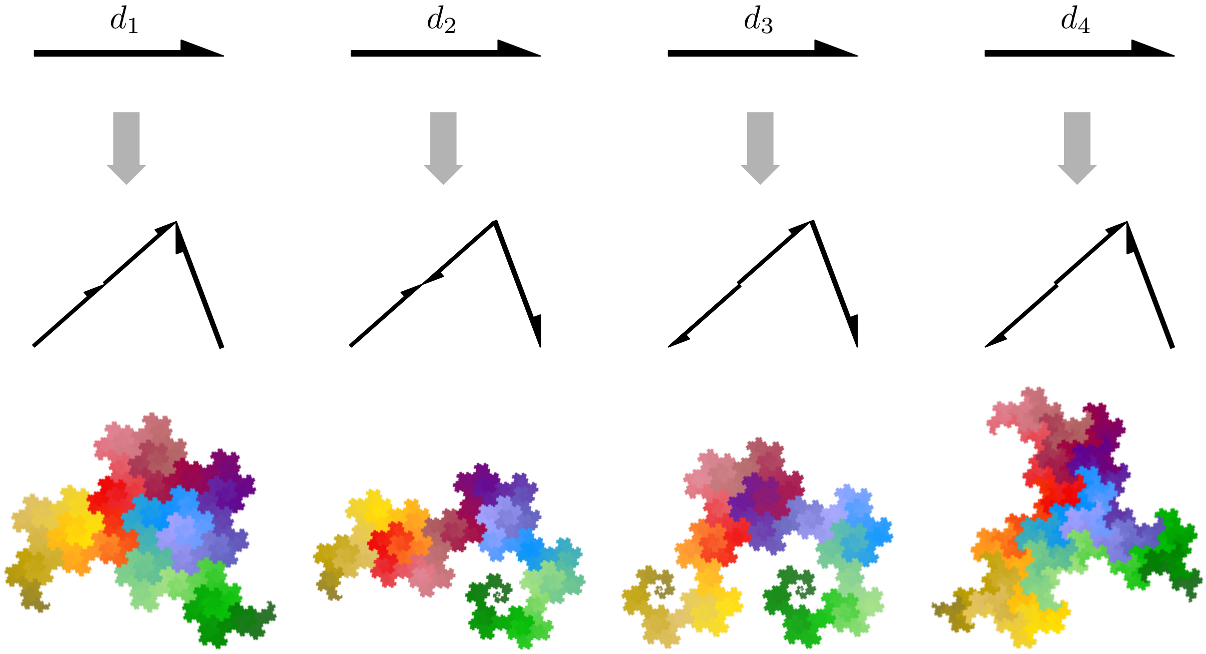

Figure 4: Definitions and colour trail visualisations of four plane-filling curves that are related to the duck curve.

We say an edge e of G_i is a *descendant* of an edge e' of G_j if e appears in the recursive expansion of e', that is, either $e = e'$, or e is a descendant of one of the edges that replace e' in G_{j+1}. An edge e' of G_j is the level-j-ancestor of an edge e in grid G_i if e is a descendant of e'. For any $j \in \{5, ..., i\}$, the level-j ancestors of the edges of E_i are all among the edges of E_j, since by definition, the ranges of the edges of E_i contain p, and since ancestors' ranges contain their descendants' ranges, they must then also contain p. It follows that for each edge e of E_i, the duck curve between the endpoints of e is part of the duck curve between the endpoints of e_0, since that duck curve contains the level-5 ancestors of all edges in E_i. Now let the *inverse* $f^{-1}(e)$ of an edge e of E_i be the interval such that $f(f^{-1}(e))$ is the duck curve between the endpoints of e. Observe that as i tends to infinity, the edges of E_i converge to p, and simultaneously $\bigcup_{e \in E_i} f^{-1}(e)$ remains non-empty and converges to a set of at most four points in $[0, 1]$. Thus, the points of $\lim_{i\to\infty} \bigcup_{e \in E_i} f^{-1}(e)$ are the values t such that $f(t) = p$, that is, these are the places where we find p on the duck curve between the endpoints of e_0.

Note that convergence of the t-values can take various forms and the pre-image of p is not necessarily a single point. As it is the limit of convergence of the union of four intervals, it is possible that the pre-image of p consists of four points, meaning that p is visited by the curve four times[2]. It is also possible that the pre-image of p is a single point, namely if out of the four intervals for any given level i, three intervals will eventually disappear from the converging set of t-values entirely and be replaced by subintervals of the fourth interval. But in any case, the pre-image of p lies in a converging set of four intervals, so it consists of up to four points (t-values).

Since the above reasoning works for any point p in the interior of P, it follows that all points in the interior of P are part of the curve's image $\mathcal{I}$. Moreover, they are not on the boundary of $\mathcal{I}$, because for any such p, there is a distance δ such that all points within distance δ from p are also in the interior of P, and therefore also in $\mathcal{I}$. This proves that the duck curve between the endpoints of e_0 fills a part of the plane with non-zero Jordan measure, and thus it is a plane-filling curve.

Related plane-filling curves

I found four plane-filling curves that are closely related to the duck curve and can be defined on the basis of the same grid—see the definitions in Figure 4. The first three of these are not too surprising:

2. I am not saying that duck curve actually contains points that are visited four times; I am only saying that my proof technique does not rule this out. At first sight, it seems that the duck curve does not actually visit any point more than three times.

they are essentially a part of the duck curve, and thus, they consist of duck curves. To be precise, when the function f from the unit interval $[0,1]$ to a subset of the plane is a curve, let $f[a,b]$ be the curve restricted to the pre-image $[a,b]$ and reparameterised to $[0,1]$, that is, $f[a,b](t) = f(a+(b-a)t)$. When two curves f and g are the same under scaling, rotation and/or translation, we will write $f \equiv g$. Now let d be the duck curve, and let d_1, d_2 and d_3 be the first three variants from Figure 4. Now we can identify largest variant curves inside the duck curve and vice versa as follows:

$$\begin{array}{ll} d_1 \equiv d[0, \frac{3}{4}] & d \equiv d_1[\frac{1}{3}, 1] \\ d_2 \equiv d[0, \frac{1}{2}] & d \equiv d_2[0, \frac{1}{2}] \\ d_3 \equiv d[\frac{1}{8}, \frac{4}{8}] & d \equiv d_3[\frac{1}{3}, \frac{2}{3}] \end{array}$$

Thus we find duck curves as the last two thirds of d_1, the first half of d_2, or the middle third of d_3.

The fourth variant ("twisted duck") however, is different. It fills a shape that appears to be half of the shape filled by the duck curve, and it consists of parts (such as the middle two quarters) that appear to fill the same shape as the duck curve—but it fills these shapes in a different way! Figure 5 illustrates the difference, showing the duck curve on the left, and the middle two quarters of the twisted duck curve (scaled and rotated to match) on the right.

Bakker also presents a generalisation of the duck curve's construction, which he calls *Titanic curves*. These are plane-filling curves in $4k-1$ parts of two different sizes on a grid of parallelograms with side length ratio $\sqrt{2k}$, for any positive natural number k.

References

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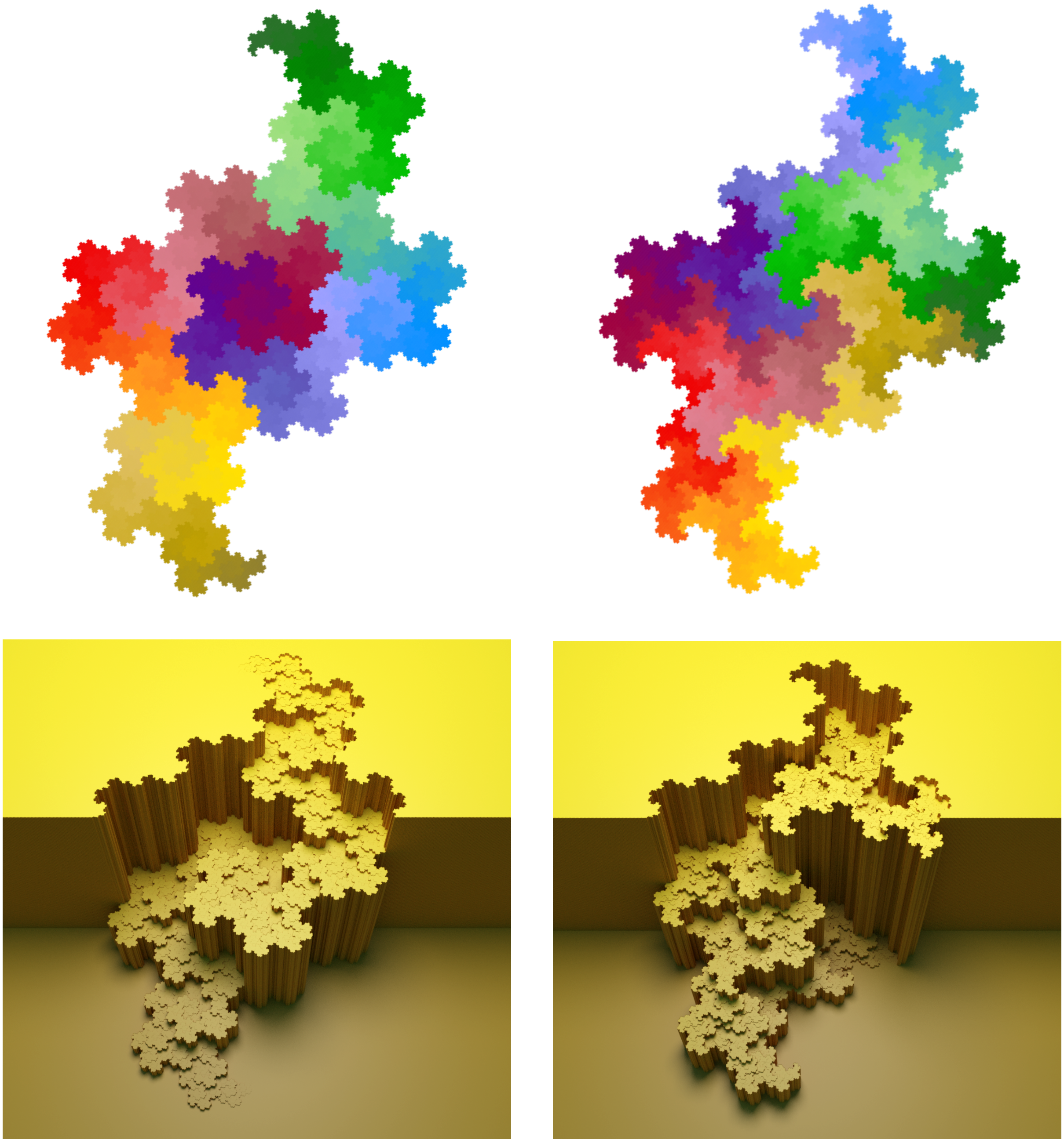

Figure 5: Colour gradient visualisations (from brown via yellow, red and blue to green) and ascending-path visualisations [7] of the duck curve (left) that results from the generator of Figure 1(b), and of the middle two quarters of the twisted duck curve (right) that results from the generator in Figure 4 (d_4).