Is Baarle random?

After our discussion over Willem's birthday vlaai, I decided to run a little experiment.

Recall the problem: suppose we have a grid of n hexagonal (or square) cells in a honeycomb (or square grid) pattern, with a roughly hexagonal (or square) outer border, and we colour each cell, independently, white with probability 1/2; if we do not colour it white, we colour it black. The resulting coloured grid may show enclaves or even enclaves within enclaves. The question is how deeply nested enclaves we can expect to find.

More precisely, let the *depth* of a cell be the minimum number of times we have to cross a white/black or black/white border to walk from that cell to the boundary of the grid (without going through vertices), and let the depth of a coloured grid be the depth of the deepest cell in the grid.

For various values of n, I generated 10 000 random colourings of a hexagonal or a square grid and I calculated:

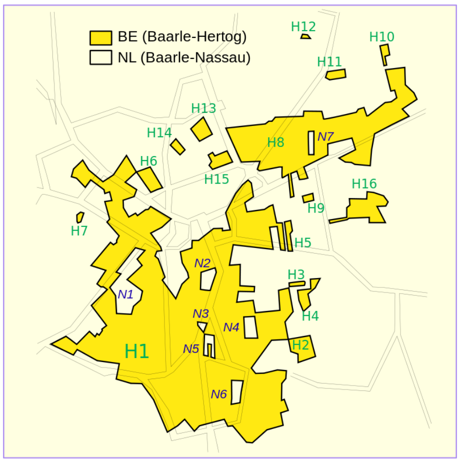

Figure 1: Baarle-Hertog and Baarle-Nassau. Source: Tos, https://commons.wikimedia.org/w/index.php?curid=7433802

- the average depth of the grid;
- the average number of cells that lies in an enclave, that is, the average number of cells that has depth larger than zero;
- the average number of enclaves.

Here is what I found:

hexagonal grid (honeycomb)					square grid				
n	$\log_{16} n$	depth	enclaved	enclaves	n	$\frac{1}{9}n^{3/7}$	depth	enclaved	enclaves
7	0.7	0.0	$0.00n$	$0.002n$	16	0.4	0.2	$0.02n$	$0.02n$
37	1.3	0.4	$0.02n$	$0.012n$	64	0.7	1.0	$0.09n$	$0.05n$
169	1.9	1.0	$0.05n$	$0.022n$	256	1.2	1.8	$0.23n$	$0.08n$
721	2.4	1.3	$0.10n$	$0.028n$	1 024	2.2	2.9	$0.45n$	$0.10n$
2 977	2.9	2.0	$0.15n$	$0.031n$	4 096	3.9	4.6	$0.67n$	$0.12n$
12 097	3.4	2.3	$0.20n$	$0.033n$	16 384	7.1	7.6	$0.82n$	$0.12n$
48 769	3.9	3.0	$0.25n$	$0.034n$	65 536	12.9	13.2	$0.91n$	$0.13n$
195 841	4.4	3.4	$0.30n$	$0.035n$	262 144	23.3	24.0	$0.95n$	$0.13n$
784 897	4.9	4.0	$0.35n$	$0.035n$	1 048 576	42.3	43.3	$0.98n$	$0.13n$

The hexagonal grids seem to have logarithmic depth, whereas the square grids seem to have polynomial depth. Does anybody know how to prove the pattern that we can see here? And how does it continue? The enclaved fraction of the hexagonal grid cannot keep growing logarithmically forever. Will it converge to 1, or to some smaller value?

From the results we learn that, if grid cells have an area of 0.3 ha (the approximate size of the smallest enclaves in Baarle), then the number of enclaves seen in Figure 1 would be obtained by a random colouring of a hexagonal grid of about 250 ha. With a square grid, the same number of enclaves would already be obtained with about 90 ha. However, the area shown in the figure is much larger (about 900 ha). This provides evidence that the division of Baarle between Belgium and The Netherlands is not entirely random.

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